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A REMARK ON THE POLYTOPALITY OF AN INTERESTING 3-SPHERE

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ABSTRACT

We give a simple proof establishing the polytopality of the 3-sphere described in [3].

In a recent paper ([3]) Bokowski, Ewald and Kleinschmidt describe a simplicial 3-sphere $B(P)$ with 10 vertices which has a combinatorial automorphism φ, such that $B(P)$ is polytopal, that is, it is realizable as the boundary complex of a 4-polytope P in the euclidean space E^4, but there is no such realization P for which φ is realized by an affine transformation of E^4. Most of that paper is devoted to the proof of the polytopality of $B(P)$.

We give here a short, self-contained and non-computational proof establishing the polytopality of $B(P)$. The proof is based on the following simple Lemma, which is a particular case of Theorem 2 in [1] and has been used in [2].

LEMMA. *Let L be an edge of a 4-polytope $Q \subset E^4$ and let $F_1, \dots, F_n$ be a chain of facets of Q around L, that is, L is an edge of each F_i, $1 \leqq i \leqq n$, and $\dim(F_i \cap F_{i+1}) = 2$ for $1 \leqq i \leqq n-1$. Then there is a point x in E^4 which lies beyond $F_1, \dots, F_n$ with respect to Q and beneath all the other facets of Q, and* $\operatorname{vert}\operatorname{conv}(Q \cup \{x\}) = \operatorname{vert} Q \cup \{x\}$.

Our terminology follows [2], in particular, Sections 4, 5.

Let S be the simplicial 3-sphere with 10 vertices described in [3] and denoted there by $B(P)$. Let S_1 be the quasisimplicial 3-sphere obtained from S by the removal of star $(1, S)$ and refilling the hole thus created with the two 3-simplices 2347, 2467 and the double tetrahedron 46825 (468 being the common base of the two tetrahedra, and 25 the missing edge) and their faces. Those three facets of S

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form a chain around the edge 24. Note that S_1 has only 9 vertices, and 25 is not an edge in antistar $(1, S)$. It follows from our Lemma that if S_1 is polytopal then so is S. (S is realized by $\operatorname{conv}[Q \cup \{x\}]$ where Q is any 4-polytope realizing S_1, $n = 3$, $F_1 = 2347$, $F_2 = 2467$, $F_3 = 46825$.)

To prove the polytopality of S_1 we remove from S_1 the complex star $(10, S_1)$ and replace it by the 3-simplices 2379, 3579 and the double tetrahedron 58936 (36 being the missing edge. Note that 36 is not in antistar $(10, S_1)$. Those 3 cells form a chain around the edge 39, and hence, using the Lemma again, it is sufficient to show that the resulting sphere S_2 (which has only 8 vertices) is polytopal.

Now, S_2 is in the list of the polytopes described in [2], and therefore it is indeed polytopal. Alternatively, remove from S_2 the complex star $(7, S_2)$ and refill the hole with the four simplices 4569, 2469, 2349 and 3459, which form a chain around the edge 49, thus obtaining a 3-sphere S_3 with just 7 vertices. Using the Lemma once again, it is sufficient to prove the polytopality of S_3. But it is well known (see [4] and the end of Section 4 in [2]) that every 3-sphere with 7 vertices is polytopal.

REFERENCES

1. A. Altshuler and I. Shemer, *Construction theorems for polytopes*, Isr. J. Math. **47** (1984), 99–110.

2. A. Altshuler and L. Steinberg, *Enumeration of the quasisimplicial 3-spheres and 4-polytopes with eight vertices*, Pac. J. Math., to appear.

3. J. Bokowski, G. Ewald and P. Kleinschmidt, *On combinatorial and affine automorphisms of polytopes*, Isr. J. Math. **47** (1984), 123–130.

4. P. Kleinschmidt, *Sphären mit wenigen Ecken*, Geom. Dedicata **5** (1976), 307–320.

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